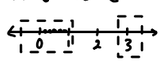
Topology:

Examples:

$A = \{1,2,3\} \cup \{1/n : n \in \mathbb{N}\}$



$der(A) = \{0\}$

$cl(A) = A \cup der(A)$

Definitions:

$B_r(x)$ (open ball) $= \{y \in \mathbb{R}^d \mid \|y-x\| < r\}$

$\bar{B}_r(x)$ (closed ball) $= \{y \in \mathbb{R}^d \mid \|y-x\| \le r\}$

$\hat{B}_r(x)$ (punctured ball) $= \{y \in \mathbb{R}^d \mid 0 < \|y-x\| < r\}$

Open set: If $\forall x \in A$, $\exists r > 0$ s.t. $B_r(x) \subseteq A$

Closed set: If A^c is open

Interior points: $x \in int(A)$ if for $A \subseteq \mathbb{R}^d$, $\exists r > 0$ s.t. $B_r(x) \subseteq A$

Cluster point: $x \in der(A)$ if $A \cap \hat{B}_r(x) \neq \emptyset$ $\forall r > 0$

Limit point: $x \in cl(A)$ if $A \cap B_r(x) \neq \emptyset$ $\forall r > 0$

Isolated point: x is isolated if $\exists r > 0$ s.t. $A \cap B_r(x) = \{x\}$

Boundary point: $x \in bd(A)$ ($\partial(A)$) if $\forall r > 0$, $B_r(x)$ has a point in A and a point in A^c

Compact: $K \subseteq \mathbb{R}^d$ is compact if it is closed or bounded.

equivalently, if every seq. in A has a subseq. that cv to a point $\in A$

Continuous: If cont. then holds for a, f is cont. at a. If holds $\forall a \in A$, f is cont.

Open covering: $\mathcal{U} \subseteq \mathbb{R}^d$ is open cover if each set in $\mathcal{U}$ is open and if $A \subseteq \bigcup_{i=1} \bigcup_{u \in \mathcal{U}} u$

Finite subcover: If $U_1, \dots, U_n$ in $\mathcal{U}$ s.t. $A \subseteq \bigcup_{i=1}^{n} U_i$

Uniform Continuous: $f: A \to \mathbb{R}$ is unicont. if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x)-f(y)| < \varepsilon$ if $\|x-y\| < \delta$ $\forall x, y \in A$

Propositions:

- A is open iff $int(A) = A$
- Union of open sets of $\mathbb{R}^d$ is open
- Intersection of finite open sets of $\mathbb{R}^d$ is open
- Union of finite closed sets of $\mathbb{R}^d$ is closed
- Union of closed sets of $\mathbb{R}^d$ is closed
- Always: $der(A) \subseteq cl(A)$
- Always: $cl(A) = A \cup der(A) = A \cup bd(A)$, but $der(A) \neq bd(A)$
- Limit point $x \in \mathbb{R}^d$ is a cluster point iff it is not isolated
- Always: $bd(A) = bd(A^c)$
- Lemma 1: If $K_1 \subseteq \mathbb{R}^m$ and $K_2 \subseteq \mathbb{R}^n$ are both compact, $K_1 \times K_2 \subseteq \mathbb{R}^{m+n}$ compact
- Lemma 2: The set $[a,b] \subseteq \mathbb{R}$ (closed/bdd 1D) satisfies HB
- Lemma 3: If $A \subseteq \mathbb{R}^d$ satisfies HB, then $B \subseteq A$ does too
- Set is open $\iff A = int(A) \iff A \cap bd(A) = \emptyset \iff \forall x, \exists r$ s.t. $B_r(x) \subseteq A$
- Set is closed $\iff der(A) \subseteq A \iff bd(A) \subseteq A \iff A = cl(A)$

Theorems:

Boundedness: TFAE:

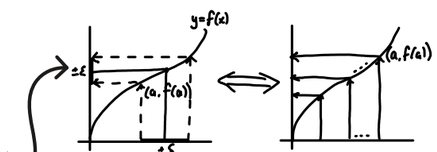
1) $\{\vec{x_n}\} \in \mathbb{R}^d$ is bounded if $\exists M > 0$ s.t. $\|\vec{x_n}\| < M$ $\forall n$
2) $\{\vec{x_n}\} \in \mathbb{R}^d$ is bounded if $\forall i \in [1,d]$, $\exists m_i \le M_i$ s.t. $m_i \le (\vec{x_n})_i \le M_i$ $\forall n$

BW: If A is bounded, infinite $\subseteq \mathbb{R}^d$, $\exists x \in \mathbb{R}^d$ s.t. $A \cap B_r(x)$ is infinite $\forall r > 0$

Words: Any infinite set that also stays within a range must "clump together" or accumulate around at least one point

Charac. of Closed Sets: TFAE:

i) A is closed
ii) $der(A) \subseteq A$
iii) $bd(A) \subseteq A$
iv) $cl(A) \subseteq A$
v) If (x_n) is any convergent sequence in A, then $\lim_{n \to \infty} x_n \in A$



Continuous Functions: $\forall a \in A$, and $f: A \to \mathbb{R}$, TFAE:

i) $\forall a \in A$ and $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(x)-f(a)| < \varepsilon$ $\forall x \in A \cap B_\delta(a)$
ii) if $(a_n) \in A$ cv to a, then $\lim_{n \to \infty} f(a_n) = f(a)$

Equivalently, $f: \mathbb{R}^d \to \mathbb{R}^m$ is cont. at $x \in \mathbb{R}^d$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall z \in \mathbb{R}^d$, $\|z-x\| < \delta \Rightarrow \|f(z)-f(x)\| < \varepsilon$

Equivalently, $z \in B_\delta(x) \Rightarrow f(z) \in B_\varepsilon(f(x))$

Preservation and IVT: $A \subseteq \mathbb{R}^d$ is compact and $f: A \to \mathbb{R}^m$ is cont.

i) $im(f) = \{f(x) \mid x \in A\}$ of f is compact $\subseteq \mathbb{R}^d$
ii) If f cont. in $[a,b]$, and L between $f(a)$ and $f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = L$

Heine-Borel: For $A \subseteq \mathbb{R}^d$, TFAE:

i) A is closed and bounded
ii) every seq. in A has subseq. cv to a point in A
iii) every open covering of A has a finite subcover

Heine-Cantor: If $K \subseteq \mathbb{R}^d$ compact, then $f: K \to \mathbb{R}$ is cont. iff unicont.

Compactness/Cont.: If $A \subseteq \mathbb{R}^d$ compact and if $f: A \to \mathbb{R}^m$ cont., then $im(f)$ is compact

Infinite Series:

Definitions:

Goal: Give a meaning to an infinite sum of real numbers
i.e. limit of a sequence of finite sums

- Infinite Series: of $a_k \in \mathbb{R}$ is the expression $\sum_{k=1}^{\infty} a_k$.
  n-partial sum: $S_n := \sum_{k=1}^{n} a_k \ (\in \mathbb{R})$
- Seq. of partial sums: $(S_n)_n \ (\in \mathbb{R})$
- Behavior:
  i) $\sum_{k=1}^{\infty} a_k$ converges if (S_n) of partial sums converges, and $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n \ (\in \mathbb{R})$
  ii) $\sum_{k=1}^{\infty} a_k$ diverges if (S_n) of partial sums diverges, and $\sum_{k=1}^{\infty} a_k$ has no value assigned $(\notin \mathbb{R})$

- Non-negative series $\sum_{1}^{\infty} a_k = (S_n)$ is nondecreasing
- General Series:
  For (a_n) positive or negative, $p_n = \begin{cases} a_n, & a_n \geq 0 \\ 0, & a_n < 0 \end{cases}$. $q_n = \begin{cases} 0, & a_n \geq 0 \\ -a_n, & a_n < 0 \end{cases}$
  $p_n, q_n > 0$, $a_n = p_n - q_n$, and $|a_n| = p_n + q_n$
- $U_n = \sum_1^n p_k$, $V_n = \sum_1^n q_k$, $S_n = U_n - V_n$ and $U_1 \leq U_2 \leq \cdots$ $V_1 \leq V_2 \leq \cdots$
- let $U = \lim_{n\to\infty} U_n$, $V = \lim_{n\to\infty} V_n$, then:
  i) $U < \infty$, $V < \infty$, then $\lim_{n\to\infty} S_n = U - V$
  ii) $U < \infty$, $V = \infty$, then $\lim_{n\to\infty} S_n = -\infty$
  iii) $U = \infty$, $V < \infty$, then $\lim_{n\to\infty} S_n = \infty$
  iv) $U = \infty$, $V = \infty$, then $\lim_{n\to\infty} S_n$ is inconclusive

General Convergences: $\sum a_n$ general is said:
i) to converge absolutely if $\sum |a_n|$ cv
ii) to converge conditionally if $\sum a_n$ cv but $\sum |a_n|$ dv

Examples:
$\sum_{1}^{\infty} 1$ dv since $S_n = 1 + \cdots + 1 = n$
So $S_n = n$ dv to ∞
- $\sum r^k$ div if $|r| \geq 1$, cv if $0 < r < 1$
  $= \frac{1 - r^{n+1}}{1 - r}$. Terms $\to 0$ not fast enough

Show that $\frac{1}{2^k - 1}$ converges
let $a_k = \frac{1}{2^k - 1}$, let $b_k = \frac{1}{2^{k-1}}$
We know $2^k - 1 \geq 2^{k-1}$, then $\frac{1}{2^k - 1} \leq \frac{1}{2^{k-1}} \Rightarrow a_k \leq b_k$
If b_k converges, a_k converges. Since b_k is geometric and ratio $1/2$, it converges thus $a_k = \frac{1}{2^k - 1}$ converges ∎

Show that $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges:
Note that $\frac{1}{n} = \int_0^1 x^{n-1} dx$
$\sum_1^m \frac{(-1)^{n+1}}{n} = \sum_1^m (-1)^{n+1} \int_0^1 x^{n-1} dx$
$= \int_0^1 \sum_1^m (-1)^{n+1} x^{n-1} dx$
$= \int_0^1 \sum_1^m (-x)^{n-1} dx$
$= \int_0^1 \frac{1 - (-x)^m}{1+x} dx$
$= \int_0^1 \frac{dx}{1+x} - \int_0^1 \frac{(-x)^m}{1+x} dx$
$= \ln 2 - (-1)^m \int_0^1 \frac{x^m}{1+x} dx$
$\int_0^1 \frac{x^m}{1+x} \leq \int_0^1 x^m = \frac{1}{m+1}$. Since $1+x \geq 1$, then $\lim_{m\to\infty} \int_0^1 \frac{x^m}{1+x} dx = 0$
$\sum \frac{(-1)^{n+1}}{n} = \lim_{m\to\infty} \sum_1^m \frac{(-1)^{n+1}}{n}$
$= \lim_{m\to\infty} (\ln 2 - (-1)^m \int_0^1 \frac{x^m}{1+x} dx)$
$= \ln 2$

Propositions:

- If a_k and b_k are the same except for a finite number of terms, then the infinite series $\sum_1^\infty a_k$ and $\sum_1^\infty b_k$ have the same behavior (cv/dv)

Lemma: let $N \in \mathbb{Z}_{>0}$. Then $\sum_1^\infty a_k$ cv $\Leftrightarrow \sum_{k=N}^\infty a_k$ cv
Pf Sketch: $S_n = \sum_1^n a_k$, $T_n = \sum_N^n a_k$, $T_n = S_n - S_{N-1}$, T_n cv $\Leftrightarrow S_n$ cv ∎

- Lemma: $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Theorems:

- Convergence of Series 5.3: let $(a_k) \in \mathbb{R}$
  i) if $\sum_1^\infty a_k$ cv, then $\lim_{k\to\infty} a_k = 0$
  ii) if a_k does not cv to 0, then $\sum_1^\infty a_k$ dv
  Note: the converse is not true ($1/k \to 0$, but $\sum_1^\infty 1/k$ dv)
- Convergence of non-negative series 5.4: let $(a_k) \in \mathbb{R}$ s.t. $a_k > 0$ $\forall k \in \mathbb{Z}_{>0}$
  $\sum_1^\infty a_k$ cv $\Leftrightarrow$ seq. of partial sums S_n is bounded
- Non-neg seq. S_n is monotone, thus cv iff it is bounded above
- Comparison Tests:
  i) If $0 \leq a_k \leq b_k$ $\forall k$ and $\sum b_k$ cv, then $\sum a_k$ cv
  ii) If $0 \leq a_k \leq b_k$ $\forall k$ and $\sum a_k$ dv, then $\sum b_k$ dv
  iii) If $\begin{cases} b_k > 0 \ \forall k \\ 0 \leq \lim a_k/b_k < \infty \\ \sum b_k \text{ cv} \end{cases}$, then $\sum a_k$ cv
  iv) If $\begin{cases} b_k > 0 \ \forall k \\ 0 < \lim a_k/b_k < \infty \\ \sum b_k \text{ div} \end{cases}$, then $\sum a_k$ div
- Integral Test:
  If f is cont., non-neg valued and nonincreasing on $[1, \infty]$,
  and satisfies $a_k = f(k)$ $\forall k \in \mathbb{Z}_{>0}$, then $\sum a_k$ and $\int_1^\infty f(x) dx$ both cv or dv
- Positive Series Ratio Test: Assume $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$ exists, then:
  i) If $L < 1$, then $\sum a_k$ cv
  ii) If $L > 1$, then $\sum a_k$ dv
  iii) If $L = 1$, inconclusive
- Positive Series Root Test: let $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$, then:
  i) If $L < 1$, then $\sum a_k$ cv
  ii) If $L > 1$, then $\sum a_k$ dv
  iii) If $L = 1$, inconclusive
- let a_n be general. If $\sum |a_n|$ cv, then $\sum a_k$ cv too
- General Series Ratio Test: a_k general (non-zero terms) and $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$ exists
  i) If $L < 1$, then $\sum a_k$ cv
  ii) If $L > 1$, then $\sum a_k$ dv
  iii) If $L = 1$, inconclusive
- General Series Root Test: let $L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. then:
  i) If $L < 1$, then $\sum a_k$ cv
  ii) If $L > 1$, then $\sum a_k$ dv
  iii) If $L = 1$, inconclusive
- Alternating Series Test: If a_n is a sequence st.:
  i) $\lim_{n\to\infty} a_n = 0$
  ii) $|a_{n+1}| < |a_n|$ $\forall n$
  iii) $a_{n+1}/a_n < 0$ $\forall n$,
  then $\sum a_n$ cv
- Permuting Terms: By permuting terms of a conditionally convergent series, we can:
  i) make it converge to any number we wish or diverge to $\pm\infty$
  ii) make its partial sums oscillate between any two numbers

Sequences of Functions:

Definitions:

- Goal: How does a sequence of functions converge to a function
- 6.1: Let $I \subseteq \mathbb{R}$ be an interval and suppose (f_n) is a sequence of real-valued fxns defined on I. Then:
  i) (f_n) converges pointwise on I if $\lim_{n\to\infty} f_n(x)$ exists $\forall x \in I$ and that
  ii) series $\sum_1^\infty f_n$ converges pointwise on I if $\sum_1^\infty f_n(x)$ converges $\forall x \in I$
- If (f_n) cv pw, define f on $I := f(x) = \lim_{n\to\infty} f_n(x)$ and write $\lim_{n\to\infty} f_n = f$ pointwise
  If $\sum_1^\infty f_n$ cv pw, define f on $I := f(x) = \sum_1^\infty f_n(x)$ and write $\sum_1^\infty f_n = f$ pointwise
- 6.6: Let $I \subseteq \mathbb{R}$ be an interval and suppose (f_n) is a sequence of real-valued fxns defined on I. Then:
  i) (f_n) converges uniformly to f on I and write $\lim_{n\to\infty} f_n = f$ uniformly on I if
  $\forall \varepsilon > 0$ $\exists N \in \mathbb{Z}_{>0}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq N$ and $\forall x \in I$
  ii) $\sum_1^\infty f_n$ converges uniformly to f on I and write $\sum_1^\infty f_n = f$ uniformly on I if $\lim_{n\to\infty} \sum_1^n f_k = f$ uniformly on I,
- Sup Norm: the quantity $\|f\| = \sup\{|f(x)| \mid x \in I\}$ is the sup norm of f on I.
- Notation: $B(I)$ is set of all bounded functions with domain I. $C(I)$ is set of continuous functions with domain I
- 6.15: Let I be an interval in $\mathbb{R}$. (i) the distance between two functions f and g on $B(I)$ is the number $\|f-g\|$.
  and (ii) a sequence (f_n) in $B(I)$ is said to converge to a function $f \in B(I)$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\|f_n - f\| < \varepsilon$ whenever $\forall n \geq N$
- 6.17: $(f_n) \in B(I)$ is uniformly Cauchy on I if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\|f_n - f_m\| < \varepsilon$ $\forall m, n \geq N$

Propositions:

- 6.2 Lemma: (f_n) on I cv pw iff:
  $\forall \varepsilon > 0$ and $\forall x \in I$, $\exists N \in \mathbb{Z}_{>0}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ $\forall n \geq N$
  Note: N is dependent on ε and x, i.e., $N(\varepsilon, x)$
- 6.2 Extended: (f_n) on I cv uniformly iff:
  $\forall \varepsilon > 0$ $\exists N \in \mathbb{Z}_{>0}$ s.t. $|\sum_1^n f_k(x) - f(x)| < \varepsilon$ $\forall n \geq N$ and $\forall x \in I$
  Note: N is dependent on ε and independent on x
- 6.12: Suppose (f_n) are C^1 on $[a,b]$, and assume f and g
  exist s.t. $\lim f_n = f$ and $\lim f_n' = g$, then $f' = g$
- 6.16: $(f_n) \in B(I)$ cv to $f \in B(I)$ iff $\lim_{n\to\infty} f_n = f$ uniformly on I
- 6.16b: $(f_n) \in B(I)$ is pointwise Cauchy if $(f_n(x))$ is Cauchy $\forall x \in I$.
  and (f_n) cv pw iff it is pw Cauchy
- 6.18: $(f_n) \in B(I)$ cv uniformly iff it is uniformly Cauchy
- 6.20: If I is closed and bounded, then every seq. in $C(I)$
  that is uniformly Cauchy converges uniformly to a function in $C(I)$

Theorems:

- 6.8: Suppose (f_n) are continuous on I, if $\exists f$ on I s.t. $\lim_{n\to\infty} f_n = f$ uniformly on I, then f is continuous on I
- 6.9: Suppose (f_n) are continuous on $[a,b]$, if $\exists f$ on $[a,b]$ s.t. $\lim_{n\to\infty} f_n = f$ uniformly on I, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$
- 6.11: Suppose (f_n) on $[a,b]$ and assume that (i) f_n is C^1 on $[a,b]$ $\forall n \in \mathbb{Z}_{>0}$ and (ii) there exists
  a function g defined on $[a,b]$ such that $\lim_{n\to\infty} f_n' = g$ uniformly on $[a,b]$, and (iii) there exists a point $c \in [a,b]$
  for which the limit $\lim_{n\to\infty} f_n(c)$ exists, then there is a function on $[a,b]$ s.t. $\lim_{n\to\infty} f_n = f$ uniformly on $[a,b]$ and $f' = g$
- 6.13 Weierstrass M-Test: Suppose (f_n) on I and $\exists (M_n)$ of numbers s.t. (i) $|f_n(x)| \leq M_n$ $\forall x \in I$ and $\forall n \in \mathbb{Z}_{>0}$,
  and (ii) series $\sum M_n$ converges, then the series $\sum f_n$ converges uniformly on I

Sequences of Functions:

Definitions:

- **Power Series:** $\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$, with sequence (a_n)
- **Radius of Convergence:** $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$. Any power series of the form $\sum_{n=0}^{\infty} a_n x^n$ should converge pointwise for $|x|<R$ and diverge for $|x|>R$

Propositions:

- **7.4:** If P is a polynomial function, then $\lim_{n\to\infty} \sqrt[n]{|P(n)|} = 1$
- **7.5:** Let $\sum a_n(x-c)^n$ have radius of cv R. Let $P(n)$, $Q(n)$ be poly. series and $b \in \mathbb{R}$
  - i) $\sum P(n)a_n(x-c)^n$ has radius of cv R
  - ii) $\sum a_n(x-c)^n / Q(n)$ has radius of cv R
  - iii) $\sum a_n b^n (x-c)^n$ has radius of cv $R/|b|$
- **7.7:** For any sequence (a_n) and any $c \in \mathbb{R}$, the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, $\sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$, $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}$ have the same radius of cv
- **7.10:** $\sum a_n(x-c)^n$ with $R>0$ is a C^∞-function on $(c-R, c+R)$, and $\frac{d^k}{dx^k}\left(\sum a_n(x-c)^n\right)\Big|_{x=c} = k! a_k \ \forall k \in \mathbb{Z}_{>0}$
- **7.11:** If $\sum a_n(x-c)^n$ has radius of cv $R>0$ and $\sum b_n(x-c)^n$ has $S>0$, and $\exists r$ s.t. $0<r\le \min\{R,S\}$, and $\sum a_n(x-c)^n = \sum b_n(x-c)^n \ \forall x \in (c-r, c+r)$, then $a_n = b_n \ \forall n \in \mathbb{Z}_{>0}$
- **Taylor Series:** Suppose f is a C^∞-function on I, and $c \in I$, then taylor series: $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$
- **Taylor Polynomial:** Partial sums $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ for $n \in \mathbb{Z}_{>0}$ is the nth taylor polynomial
- **Approximation Error:** From above, $f^{(k)}(c) = P_n^{(k)}(c)$. Let R_n s.t. $f = P_n + R_n$, then R_n is the error wrt P_n as an approx. of f. We want to find R_n the smallest, so $\lim_{n\to\infty} P_n(x) = f(x)$ is the question whether $\lim_{n\to\infty} R_n(x) = 0$

Theorems:

- **7.3:** Consider $\sum_{n=0}^{\infty} a_n(x-c)^n$ and the quantity $R = 1/\limsup_{n\to\infty} \sqrt[n]{|a_n|}$. where $1/0 = \infty$ and $1/\infty = 0$
  - i) if $R=0$, then the power series converges only for $x=c$
  - ii) if $0<R<\infty$, then the power series converges absolutely $\forall x$ s.t. $|x-c|<R$, diverges $\forall x$ s.t. $|x-c|>R$, and converges uniformly on $[c-r, c+r] \ \forall r \in (0,R)$
  - iii) If $R=\infty$, then the power series converges absolutely $\forall x$ and uniformly on any interval of finite length
- **7.6:** A power series $\sum a_n(x-c)^n$ whose radius of cv R is positive is a continuous function on $(c-R, c+R)$
- **7.8:** If $\sum_{n=0}^{\infty} a_n(x-c)^n$ has $R>0$, then:
  - i) $\int_c^x \left(\sum a_n(t-c)^n\right) dt = \sum \frac{a_n}{n+1}(x-c)^{n+1}$, and
  - ii) $\frac{d}{dx}\left(\sum a_n(x-c)^n\right) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \ \forall x \in (c-R, c+R)$
- **7.12:** Suppose f on I, $c \in I$, and $n>0$. If $f^{(n+1)}$ exists and is continuous on I and if R_n is as above, then $R_n(x) = \frac{1}{n!}\int_c^x (x-u)^n f^{(n+1)}(u)\,du$ and $\exists \xi \in I$ s.t. $R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$